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Diakoptics

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DIAKOPTICS

by

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Introduction

The analysis of a large system to determine its response to applied forcing functions involves the solution of a set of simultaneous equations having many variables. The solution of such a set of equations of, for example, eight variables and having real-number coefficients is a formidable task using a slide rule, and even with a medium-size computer the solution of fifty equations requires considerable time and effort. If the coefficients are complex numbers, the effort is increased four-fold; if they are polynomials such as might be obtained from transformed systems, very severe limits are placed on the size of the system.

Diakoptics is a method of determining the response of a system by breaking the system into parts and solving several sets of simultaneous equations each having fewer variables than the original set.

This thesis represents an investigation of the method of Diakoptics as originated by Gabriel Kron and published in The Electrical Journal (London)[1] in 1957 and 1958. The investigation is carried out with respect to electrical systems, although Kron extends it to other types of systems. In addition, Diakoptics is applied specifically to the

[1]See Bibliography

inversion of matrices which are not necessarily derived from electrical networks.

A basic knowledge of the methods of analysis of electrical networks in the steady state is assumed of the reader. Also, rudiments of matrix algebra are assumed.

I. Basic Principles

Diakoptics, of Kron's Theory of Tearing[1], has as its basis the tearing apart of a network into smaller sub-networks. These sub-networks are individually solved by inverting their impedance or admittance matrices, and the solution to the original network established with the aid of these sub-network solutions and an "intersection network" which takes into account the tie-lines which originally connected the sub-networks together.

In terms of a network which is to be solved on a node basis, tearing consists of dividing the network into suitable sub-networks and removing from the network the branches which cross the division line. Thus, in Figure 1, a network is divided into two parts by the dividing line

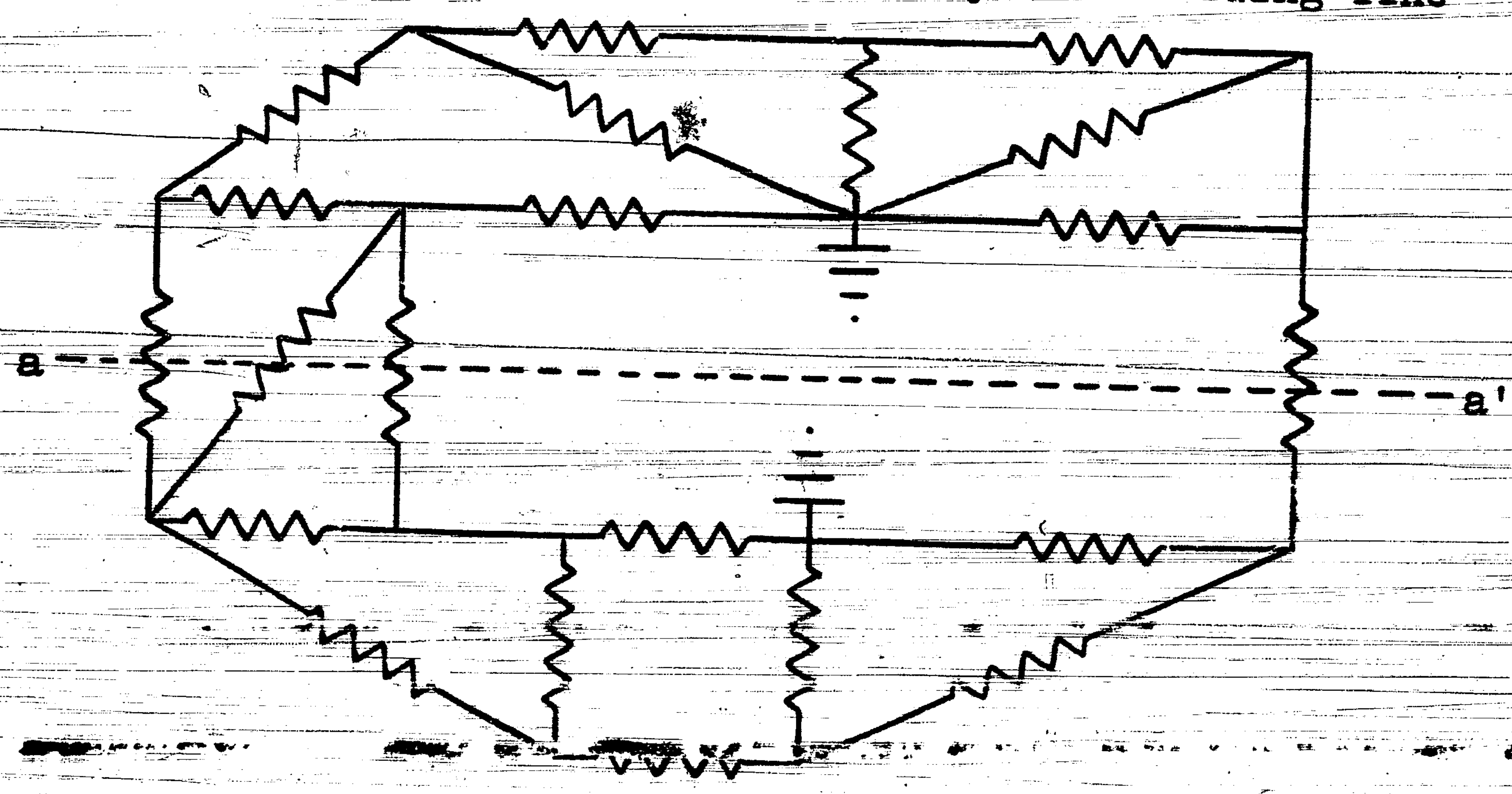
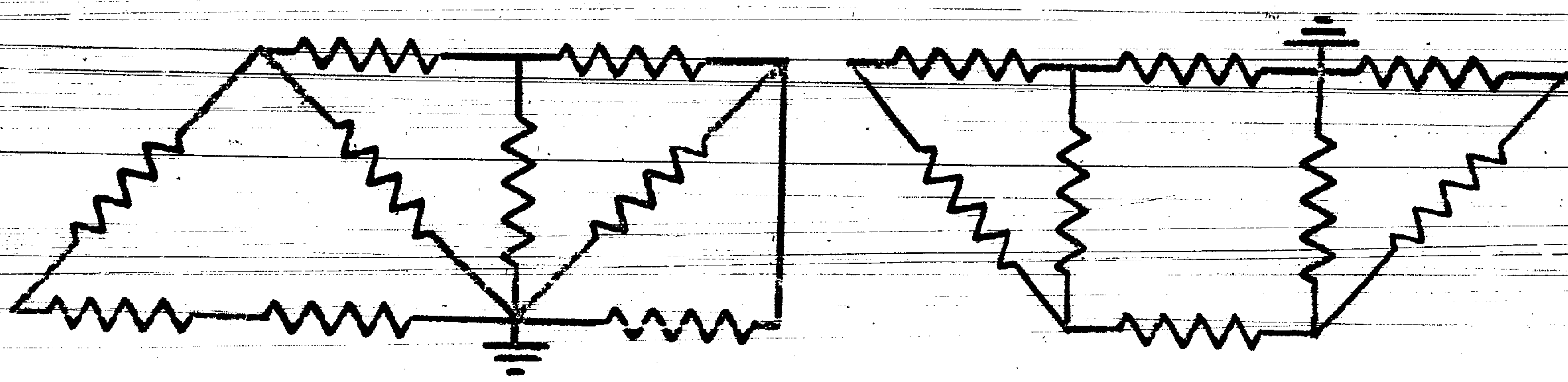


Figure 1. Line of Tearing.

a-a'. The branches which cross this dividing line are removed from the network to obtain the two sub-networks shown in Figure 2. The branches which are removed from



A B

Figure 2. The Sub-Networks.

the network are set aside for later use in the intersection network. The choice of dividing lines is governed by two considerations: first, the sub-networks which result must be small enough for convenient solution by the method available; and second, it is desirable to remove a minimum number of branches from the network. There may be as many lines of tearing as necessary to fulfill these conditions.

The sub-networks are solved by finding their admittance matrices and inverting them. The resultant matrices are then combined into a single impedance matrix.

The intersection network is formed from the branches which were removed from the original network in the tearing of the network and portions of the solutions of the sub-networks. These are shown in Figure 3. The "black boxes" marked A and B contain portions of the sub-networks which

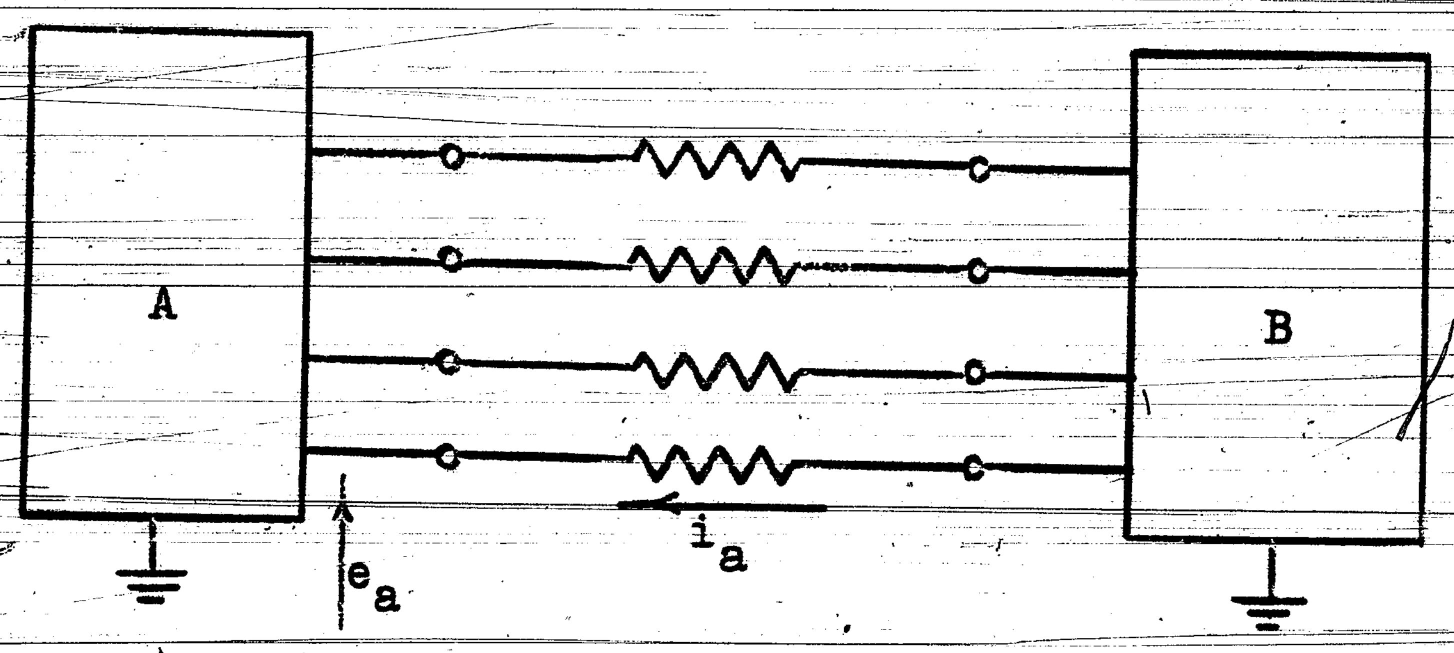


Figure 3. The Intersection Network.

are to be retained - namely, the portions which connect to the tie-lines. In terms of the sub-networks, the solutions which were obtained can be considered to represent a mesh network. Thus, if a current i_a (Figure 3) is impressed at a node, a voltage e_a appears at the node. Voltages may be summed around the loops formed by the sub-networks, tie lines, and ground. This summation is performed by setting up a connection matrix which shows how the currents in the tie-lines are impressed on the sub-networks. The sub-network solutions are multiplied by the connection matrix twice in order to relate the sub-networks to the loops of the intersection network, and the impedances of the tie-lines added to the result.

The resultant impedance matrix then may be said to be the intersection network impedance. The intersection network then is solved by inverting the impedance matrix, and the solution combined with the connection matrix and the sub-network solution matrices to form the complete solution for the original network.

A mesh network is solved in the same manner; all quantities and operations of the node network become the respective duals for the mesh network.

The inversion of matrices is limited to those which are symmetrical. For such matrices it is possible to obtain a network (or graph) which could be represented by the matrix. (This graph, for a numerical matrix, might be of the form of an electrical resistance network with both positive and negative resistors; in any event, the network can be torn and the solution obtained as in the case of an ordinary electrical network.) However, it is not necessary to draw the graph; the original matrix can be inverted by the same method without actually drawing the network. This method of solution is not the same as the method of partitioning, but rather makes use of the zero elements of the matrix in arriving at a solution more easily than by direct inversion.

The fact that Diakoptics makes use of the zero elements of a matrix in the process of solution is significant. It is characteristic of the equations of many physical systems that a number of the coefficients will be zero. For example, in an electrical network, a given node is connected to only a limited number of other nodes. Thus, their matrix representations will have a number of zero elements. It is this property which allows the extensive simplifications here described.

II. Solution of Networks

In order to completely describe the solution of networks using the principles of Diakoptics, the solution will be described in detail. The solution may be based on either a node (or admittance) analysis, or a mesh (or impedance) analysis. In either event, the same steps are carried out in the analysis. In order to clarify the solution of problems in terms of Diakoptics, node networks will be discussed in detail; then the method will be extended to mesh networks. Finally it will be shown that matrices can be inverted in pieces without using the graph of the system under consideration.

A. Node Networks

The solution of networks excited by current sources involves, in general, the solution of a set of equations of the form $I = YE$. (If the network also contains voltage sources, these may be replaced by their Norton equivalents.) The solution utilizing the concepts of Diakoptics can be outlined in six steps:

1. The network is torn into sub-networks which are as large as convenient. These sub-networks are solved on a nodal basis by inverting the admittance matrix for each sub-network. The inverse matrices are arranged along the diagonal

of a large matrix Z_1.

2. The intersection network is established. The impedances of the cut branches which were removed in the tearing process are arranged in a matrix $\underline{z}$. This matrix will be a diagonal matrix unless mutual couplings exist between the elements of the cut branches.

3. The intersection network is represented by a connection matrix C, whose elements are 1, -1, or 0.

4. The sub-network solutions of Z_1 are related to the intersection network using C by the transformation law[3] $Z_1' = C_t Z_1 C$. The resultant matrix Z_1' is added to the matrix $\underline{z}$ to form the impedance matrix representing the intersection network by $Z_2 = Z_1' + z$.

5. The intersection network is solved by inverting Z_2, or $(Z_2)^{-1} = Y_2$.

6. Z_1, C, and Y_2, along with the matrix of impressed currents I, form the complete solution for the network:

$$E = Z_1 I - Z_1 C Y_2 C_t Z_1 I \qquad (1)$$

In order to identify each of the above steps, an example will be worked out. Consider, then, the network of Figure 4. Following the outline given above, the

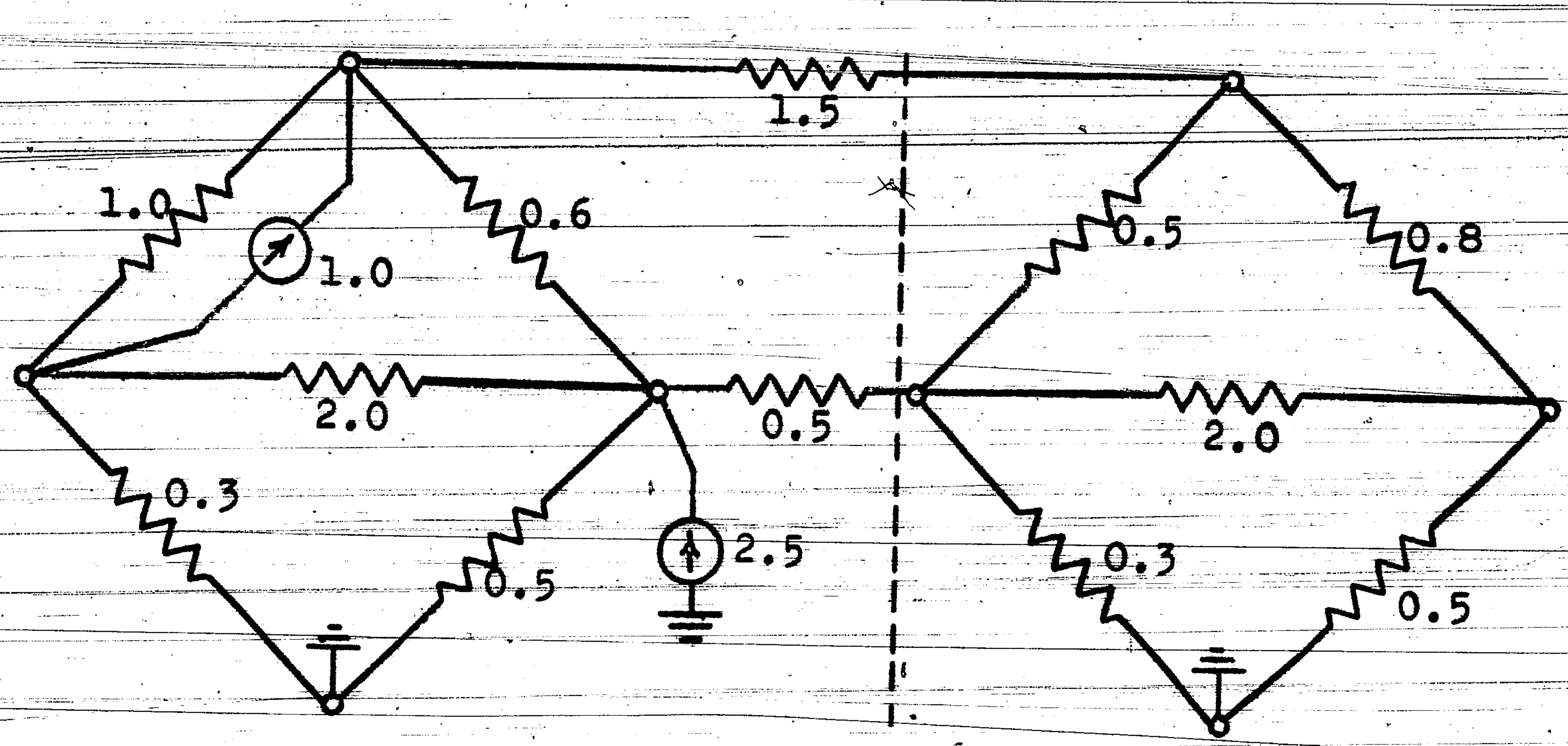


Figure 4. Node Network (admittances shown).

stepwise solution proceeds as follows:

1. Tear the network and solve the sub-networks. The network of Figure 4 is torn into two sub-networks as indicated by the dotted line; the cut branches removed. The two resulting sub-networks are shown in Figure 5.

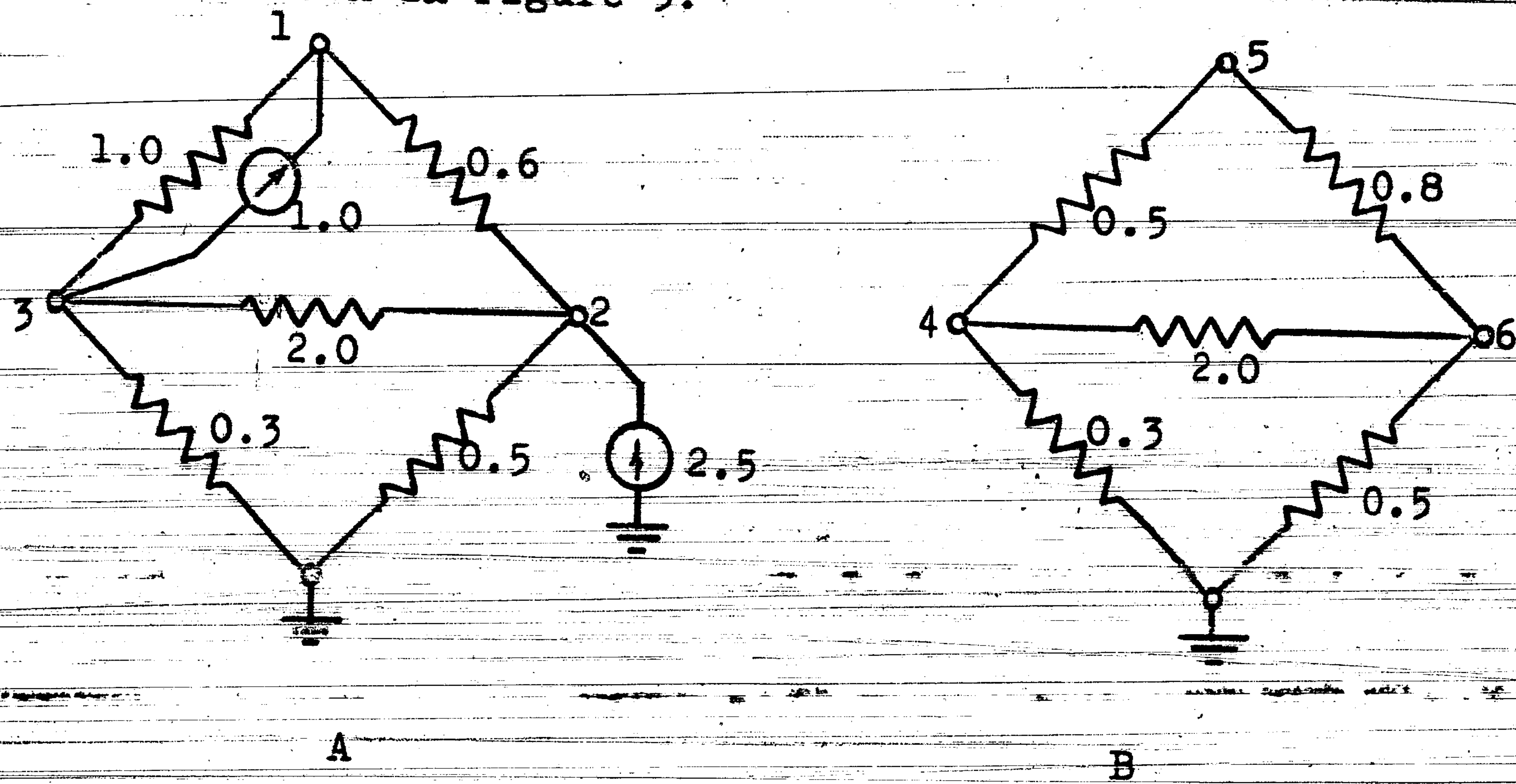


Figure 5. The Sub-Networks.

Note that although the removal of the cut branches would ordinarily allow the second network to be considered a single node-pair network, in this case the original nodes must be retained.

The two sub-networks are now solved by any convenient means to determine the inverses of their admittance matrices. In matrix form, the node equations of sub-network A are:

$$\begin{array}{c|c|} 1 & 1.0 \\ 2 & 2.5 \\ 3 & -1.0 \end{array} = \begin{array}{c|c|c|c|} & 1 & 2 & 3 \\ \hline 1 & 1.6 & -0.6 & -1.0 \\ 2 & -0.6 & 3.1 & -2.0 \\ 3 & -1.0 & -2.0 & 3.3 \end{array} \cdot \begin{array}{c|c|} 1 & e_1 \\ 2 & e_2 \\ 3 & e_3 \end{array} \qquad (2)$$

The solution to this equation is obtained by inverting the admittance matrix:

$$Y_a = \begin{array}{c|c|c|c|} & 1 & 2 & 3 \\ \hline 1 & 1.6 & -0.6 & -1.0 \\ 2 & -0.6 & 3.1 & -2.0 \\ 3 & -1.0 & -2.0 & 3.3 \end{array} \qquad (3)$$

$$(Y_a)^{-1} = \begin{array}{c|c|c|c|} & 1 & 2 & 3 \\ \hline 1 & 1.899 & 1.213 & 1.311 \\ 2 & 1.213 & 1.305 & 1.159 \\ 3 & 1.311 & 1.159 & 1.402 \end{array} \qquad (4)$$

Then,

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1.899 & 1.213 & 1.311 \\ 2 & 1.213 & 1.305 & 1.159 \\ 3 & 1.311 & 1.159 & 1.402 \end{array} \cdot \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 1.0 \\ 2.5 \\ -1.0 \end{bmatrix} \qquad (5)$$

The second sub-network is solved in the same manner:

$$\begin{array}{c} 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{array}{c|ccc} & 4 & 5 & 6 \\ \hline 4 & 2.8 & -0.5 & -2.0 \\ 5 & -0.5 & 1.3 & -0.8 \\ 6 & -2.0 & -0.8 & 3.3 \end{array} \cdot \begin{array}{c} 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} e_4 \\ e_5 \\ e_6 \end{bmatrix} \qquad (6)$$

$$\begin{array}{c} 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} e_4 \\ e_5 \\ e_6 \end{bmatrix} = \begin{array}{c} 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 1.407 & 1.252 & 1.156 \\ 1.252 & 2.019 & 1.249 \\ 1.156 & 1.249 & 1.306 \end{bmatrix} \cdot \begin{array}{c} 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (7)$$

$$Z_b = \begin{array}{c|ccc} & 4 & 5 & 6 \\ \hline 4 & 1.407 & 1.252 & 1.156 \\ 5 & 1.252 & 2.019 & 1.249 \\ 6 & 1.156 & 1.249 & 1.306 \end{array} \qquad (8)$$

The first step of the solution is completed by arranging these solutions along the diagonal of a large matrix Z_1:

$$
Z_1 = \begin{array}{c|c|c|}
 & a & b \\
\hline
a & Z_a & \\
\hline
b & & Z_b \\
\hline
\end{array}
$$

$$
= \begin{array}{c|c|c|c|c|c|c|}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 1.899 & 1.213 & 1.311 & & & \\
\hline
2 & 1.213 & 1.305 & 1.159 & & & \\
\hline
3 & 1.311 & 1.159 & 1.402 & & & \\
\hline
4 & & & & 1.407 & 1.252 & 1.156 \\
\hline
5 & & & & 1.252 & 2.019 & 1.249 \\
\hline
6 & & & & 1.156 & 1.249 & 1.306 \\
\hline
\end{array} \qquad (9)
$$

2. The intersection network is now established. The branches which were cut and removed in the tearing process are now placed between their terminal nodes which are obtained from the sub-networks. No other parts of the sub-networks are needed to determine the solution. These branches are shown in Figure 6.

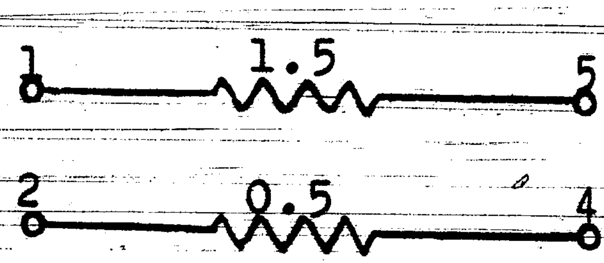


Figure 6. The Cut Branches.

As described in Section I, this network is to be solved on a mesh basis; consequently mesh currents must be established. The mesh currents flow in loops each formed from a branch of the intersection network, the sub-networks, and ground. They are shown in Figure 7. These

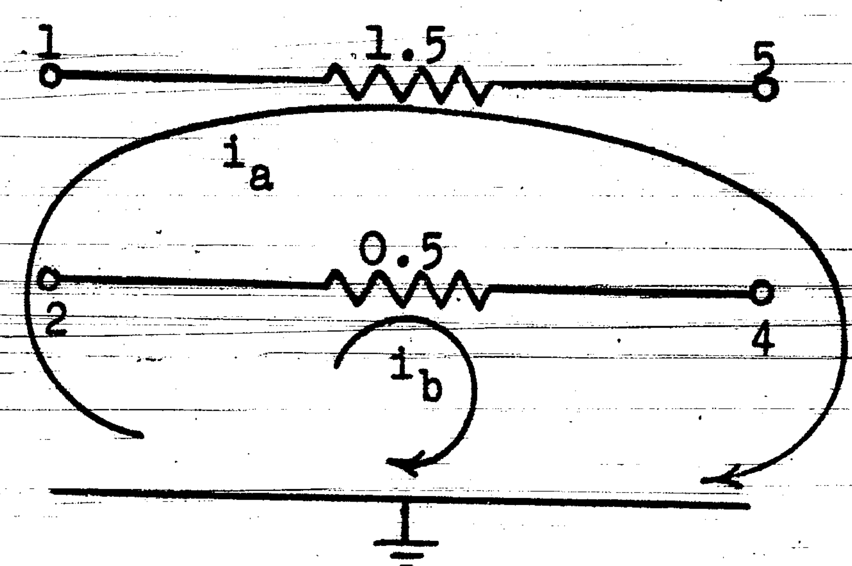


Figure 7. Loop Currents.

meshes then make it possible to establish the intersection impedance matrix $\underline{z}$, which is a diagonal matrix (but may not be if mutual impedances exist between branches of the intersection network):

$$
z = \begin{array}{c|cc} & a & b \\ \hline a & 0.667 & \\ b & & 2.000 \end{array} \qquad (10)
$$

3. The connection matrix C may also be established with the aid of the intersection network of Figure 7. The dimensions of this matrix are in the vertical direction the same as Z_1, and horizontally the same as $\underline{z}$. It relates the node currents of the sub-networks to the mesh currents

of the intersection network. Its elements are determined by the following rules:

a. If an intersection current (such as i_b) **comes out** of a sub-network node (such as 2), a 1 is placed in the mesh-current column and node row (b-column and 2-row) of C.

b. If an intersection current (such as i_a) **enters** a sub-network node (such as 5), a -1 is placed in the mesh-current column and node row (a-column and 5-row) of C.

c. If neither of the above occur, the element of C is zero.

Thus, the connection matrix for the example becomes:

$$C = \begin{array}{c|cc|} & a & b \\ \hline 1 & -1 & \\ 2 & & -1 \\ 3 & & \\ 4 & & 1 \\ 5 & 1 & \\ 6 & & \\ \hline \end{array} \qquad (11)$$

4. The matrix Z_1 is now transformed by

$$Z_1' = C_t Z_1 C \qquad (12)$$

The result of this transformation is

$$Z_1' = \begin{array}{c|cc} & a & b \\ \hline a & 3.918 & 2.466 \\ b & 2.466 & 2.711 \end{array} \quad (13)$$

The intersection impedance Z_2 is given by the sum of $\underline{z}$ and Z_1':

$$Z_2 = \begin{array}{c|cc} & a & b \\ \hline a & 4.585 & 2.466 \\ b & 2.466 & 4.711 \end{array} \quad (14)$$

This is the impedance of the intersection network, a mesh network consisting of the cut branches of the original network and the transformed subnetworks.

5. The solution of the intersection network is found by inverting Z_2:

$$Y_2 = (Z_2)^{-1} = \begin{array}{c|cc} & a & b \\ \hline a & 0.304 & -0.159 \\ b & -0.159 & 0.295 \end{array} \quad (15)$$

6. At this point the solution of the network may be considered to have been determined, since the matrices C, X_1, and Y_2 have been found. These three matrices together constitute what Kron calls the "Factorized Inverse" of the admittance matrix of the original network.[3] The solution for the voltages which appear at the nodes in

response to impressed currents were given as

$$E = Z_1 I - Z_1 C Y_2 C_t Z_1 I \qquad (1)$$

The evaluation of this equation is most easily carried out in steps such that all products are of the form of a two-way matrix multiplied by a one-way matrix:

a. $E_1 = Z_1 I$ (16)

From (2) and (6), the impressed currents are:

$I =$ (17)

1	1.0
2	2.5
3	-1.0
4	
5	
6	

Then,

$E_1 =$ (18)

1	3.622
2	3.317
3	2.805
4	
5	
6	

b. $e = -C_t E_1$

$$= \begin{array}{c|c|} \cline{2-2} a & 3.622 \\ \cline{2-2} b & 3.317 \\ \cline{2-2} \end{array} \tag{19}$$

c. $i = Y_2 e$

$$= \begin{array}{c|c|} \cline{2-2} a & 0.572 \\ \cline{2-2} b & 0.404 \\ \cline{2-2} \end{array} \tag{20}$$

d. $I' = Ci$

$$= \begin{array}{c|c|} \cline{2-2} 1 & -0.572 \\ \cline{2-2} 2 & -0.404 \\ \cline{2-2} 3 & \\ \cline{2-2} 4 & 0.404 \\ \cline{2-2} 5 & 0.572 \\ \cline{2-2} 6 & \\ \cline{2-2} \end{array} \tag{21}$$

e. $E_2 = Z_1 i'$

$$= \begin{array}{c|c|} \cline{2-2} 1 & -1.578 \\ \cline{2-2} 2 & -1.222 \\ \cline{2-2} 3 & -1.219 \\ \cline{2-2} 4 & 1.286 \\ \cline{2-2} 5 & 1.662 \\ \cline{2-2} 6 & 1.182 \\ \cline{2-2} \end{array} \tag{22}$$

f. $E = E_1 + E_2$

$$= \begin{array}{c|c|} 1 & 2.044 \\ 2 & 2.095 \\ 3 & 1.586 \\ 4 & 1.286 \\ 5 & 1.662 \\ 6 & 1.182 \end{array} \qquad (23)$$

This completes the solution.

The inverse of the complete admittance matrix Y for the original circuit may also be found. From (1),

$$Z = (Y)^{-1} = Z_1 - Z_1CY_2C_tZ_1 \qquad (24)$$

This expression may be evaluated in a manner similar to that shown above. The result is:

Z = (25)

	1	2	3	4	5	6
1	1.103	0.674	0.742	0.560	0.846	0.545
2	0.674	0.858	0.724	0.473	0.566	0.424
3	0.742	0.724	0.967	0.456	0.600	0.422
4	0.560	0.473	0.456	0.906	0.665	0.710
5	0.846	0.566	0.600	0.665	1.122	0.675
6	0.545	0.424	0.422	0.710	0.675	0.897

In determining the response voltages of (23), it was not necessary to determine the matrix Z of (25). In fact, it is considerably less work to multiply the matrices together in the fashion of (18) through (23) than to determine Z, since the latter involves multiplying two-way matrices by two-way matrices, whereas determining I requires multiplying two-way matrices by one-way matrices. Also, it is seldom necessary to have Z available, since C, Y_1, and Z_2 completely represent the system and its solution.

Additional saving of effort can be realized in large networks if the tearing can be performed in such a manner that identical sub-networks are formed, since solving one of several identical sub-networks will suffice for all of them. Also, since the solution of a sub-network places no limitations on how it is connected into the original network except that the nodes of the original network must be included in the sub-network, the solution of a sub-network may be saved for use in an entirely different network which includes the sub-network.

In networks having magnetic coupling between elements, some care must be excercised with respect to the tearing. The analysis becomes considerably more difficult[4] if mutual coupling exists between sub-networks, or between a sub-network and the intersection network. Branches of the intersection network may be coupled, however.

With reference to Step 5 of the analysis, if only the response voltages are required, it is not necessary to obtain the complete inverse of Z_2, but rather, it is required only to solve a set of simultaneous equations represented in matrix form as

$$e = Z_2 i$$

where e and i are defined by (19) and (20), respectively.

B. Mesh Networks

The analysis of mesh networks is the same as that of node networks except that every quantity is replaced by its dual. The network of Figure 8 will serve as an example.

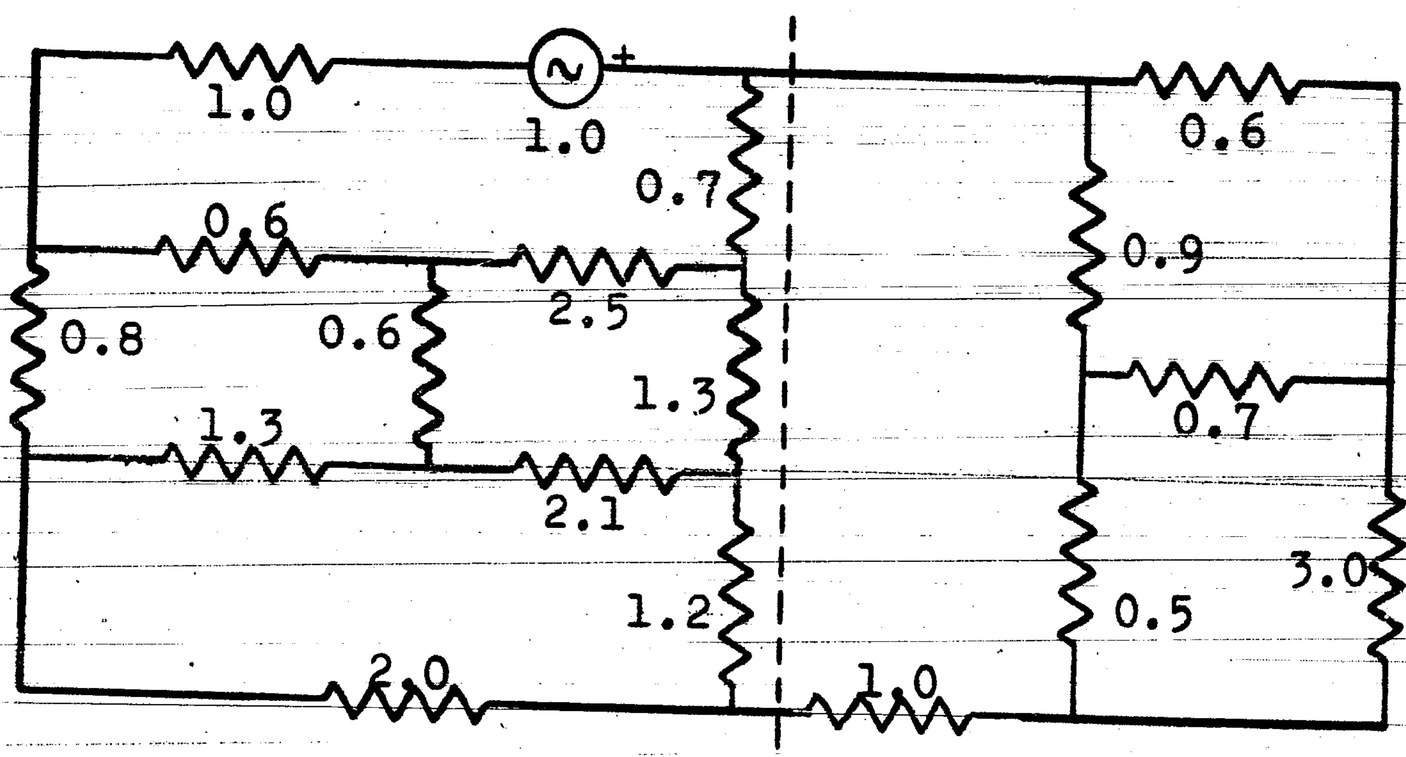
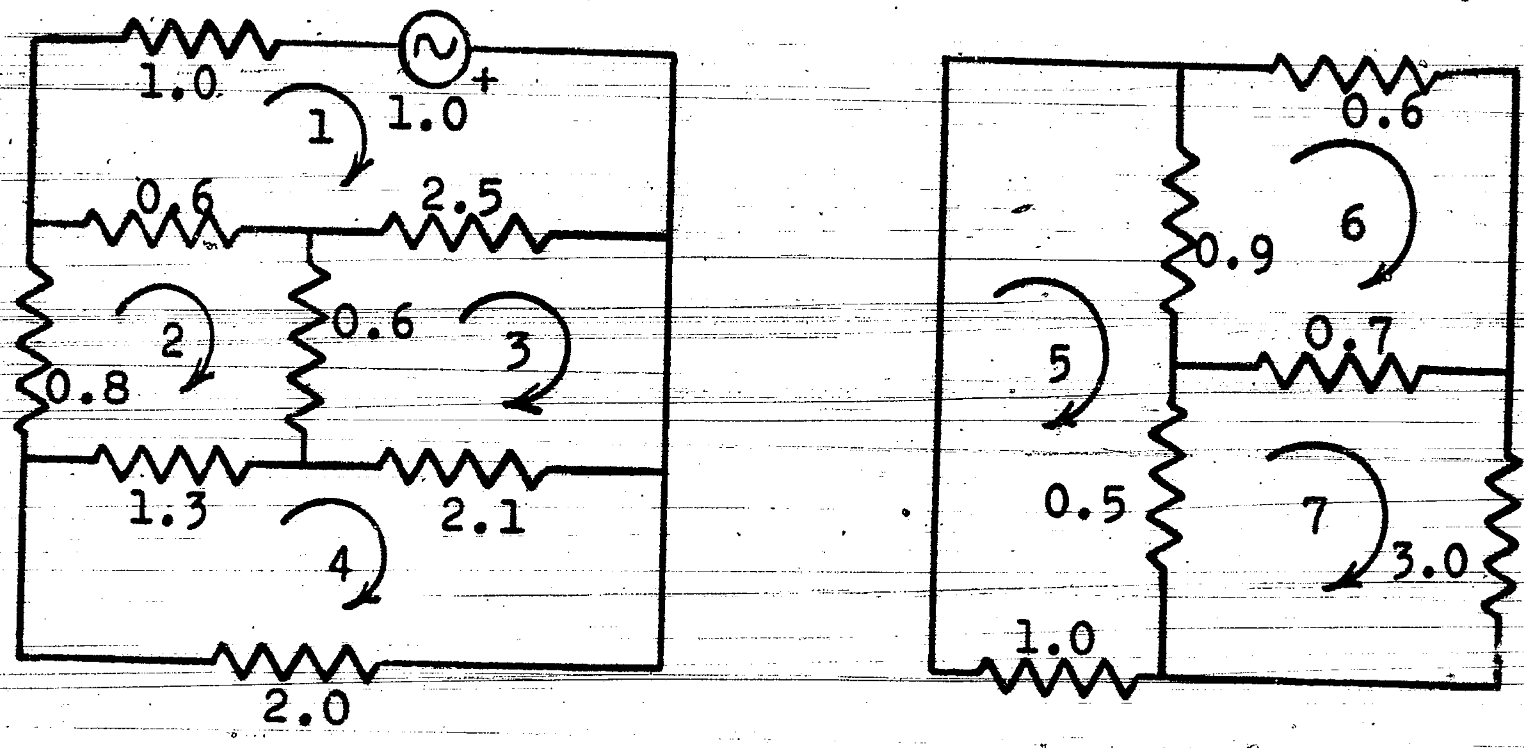


Figure 8. Mesh Network (impedances shown).

The step-by-step procedure is:

1. A mesh network is torn by short-circuiting the branches which tie the sub-networks together, and removing the impedances along the line of tearing. Thus, if the network of Figure 8 is torn along the dotted line, three impedances are

removed, and two sub-networks are formed as in Figure 9. These sub-networks are solved by



A B

Figure 9. The Sub-Networks.

obtaining and inverting their impedance matrices; the inverse matrices are then arranged along the diagonal of a large matrix Y_1:

Z_a = (27)

	1	2	3	4
1	4.1	-0.6	-2.5	
2	-0.6	3.3	-0.6	-1.3
3	-2.5	-0.6	5.2	-2.1
4		-1.3	-2.1	5.4

Z_b = (28)

	5	6	7
5	2.4	-0.9	-0.5
6	-0.9	2.2	-0.7
7	-0.5	-0.7	4.2

Y_1 =

	1	2	3	4	5	6	7
1	0.473	0.215	0.324	0.178			
2	0.215	0.467	0.240	0.206			
3	0.324	0.240	0.473	0.242			
4	0.178	0.206	0.242	0.329			
5					0.533	0.252	0.105
6					0.252	0.599	0.130
7					0.105	0.130	0.272

(29)

2. In order to establish the intersection network, the impedances which were removed in the tearing of the network are placed between the meshes which adjoined them in the original network. This is shown in Figure 10. This network is to

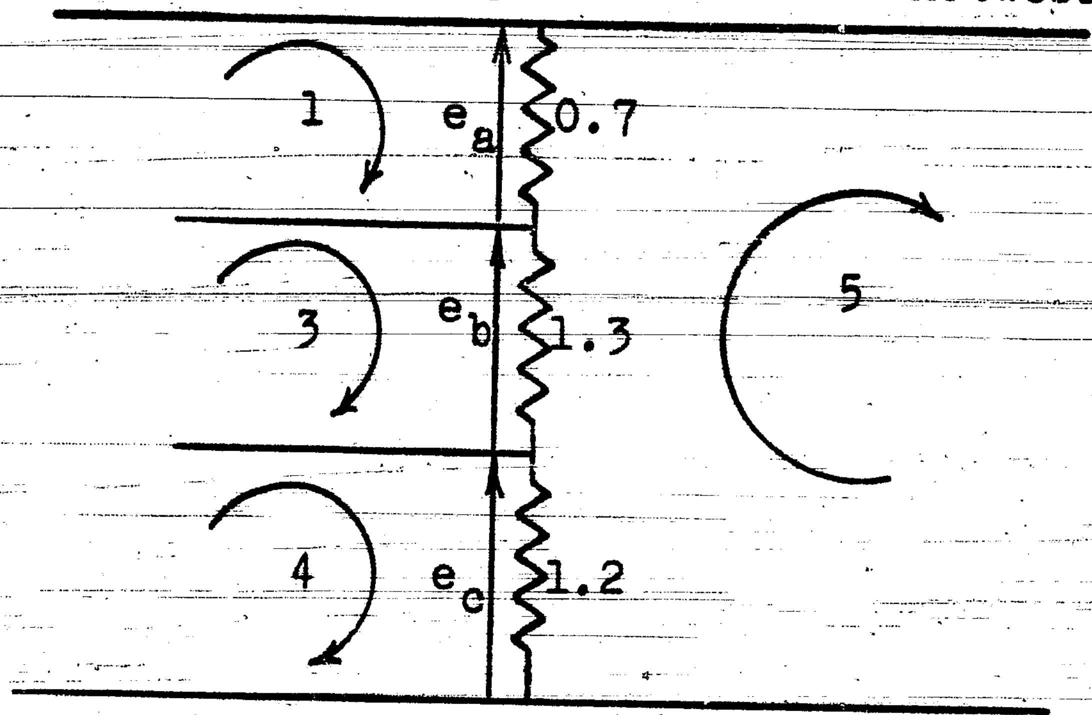


Figure 10

be solved on a node basis. Just as in the case of node analysis the currents of the intersection network were considered to be impressed upon the sub-network nodes which terminated them, so also in this case of a mesh network the voltages which appear across the impedances may be considered to be impressed in the loops of the sub-networks. These voltages are shown in Figure 10. The matrix y is now easily determined to be

$$y = \begin{array}{c|ccc} & a & b & c \\ \hline a & 1.429 & & \\ b & & 0.769 & \\ c & & & 0.833 \end{array} \qquad (30)$$

3. Figure 10 also helps to determine the connection matrix C, which is formed according to the following rules:

   a. If an intersection voltage opposes a current in an adjoining mesh, a 1 is placed in the intersection-voltage column and mesh-current row.

   b. If an intersection voltage is in the same direction as the current in an adjoining mesh, a -1 is placed in the corresponding intersection-voltage column and mesh-current row.

   c. If an intersection-voltage does not adjoin a mesh-current, a zero is placed in the corresponding element of C.

The connection matrix for the example is

$$C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc} 1 & & \\ & & \\ & 1 & \\ & & 1 \\ -1 & -1 & -1 \\ & & \\ & & \end{array}\right] \end{array} \tag{31}$$

4. From this point the solution follows exactly as in the node network. Thus,

$$Y_1' = C_t Y_1 C \tag{32}$$

$$Y_2 = Y_1 + y = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc} 2.343 & 0.857 & 0.711 \\ 0.857 & 1.776 & 0.775 \\ 0.711 & 0.775 & 1.695 \end{array}\right] \end{array} \tag{33}$$

5. $Z_2 = (Y_2)^{-1}$ (34)

6. $I = Y_1E - Y_1CZ_2C_tY_1E$ (35)

This equation is worked out in the same manner as for the previous example. The exciting voltages from Figure 8 and the resultant currents are:

$$
E = \begin{array}{c|c|}
1 & 1.0 \\
2 & \\
3 & \\
4 & \\
5 & \\
6 & \\
7 &
\end{array}
\tag{36}
$$

$$
I = \begin{array}{c|c|}
1 & 0.364 \\
2 & 0.157 \\
3 & 0.223 \\
4 & 0.127 \\
5 & 0.137 \\
6 & 0.065 \\
7 & 0.027
\end{array}
\tag{37}
$$

The inverse of the impedance matrix for the original network becomes

Y =

	1	2	3	4	5	6	7
1	0.364	0.157	0.223	0.127	0.137	0.065	0.027
2	0.157	0.424	0.170	0.156	0.102	0.048	0.020
3	0.223	0.170	0.342	0.171	0.159	0.075	0.031
4	0.127	0.156	0.171	0.259	0.123	0.058	0.024
5	0.137	0.102	0.159	0.123	0.286	0.135	0.056
6	0.065	0.048	0.075	0.058	0.135	0.544	0.107
7	0.027	0.020	0.031	0.024	0.056	0.107	0.263

(38)

C. Matrix Inversion

Kron's point of view with respect to Diakoptics is that the tearing process should combine the properties of the system graph and the matrices representing the system;[5] he does not allow for the possibility of tearing a matrix without using a graph. However, it is possible to invert matrices using Diakoptics in a straight-forward manner without reference to a graph. The analysis proceeds in the same manner as in the case of circuit analysis, except that the matrix is used as the starting point rather than the circuit. Another example will be used to clarify the procedure. Let it be desired to invert the following matrix:

$$Y = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 6 & & 4 & & \\ 2 & & 2 & 1 & 2 & \\ 3 & 4 & 1 & 3 & 3 & 6 \\ 4 & & 2 & 3 & 3 & 2 \\ 5 & & & 6 & 2 & 5 \end{array} \qquad (39)$$

The step-by-step procedure for inverting a matrix follows the outline used for network solutions; however, the steps are carried out differently in some cases:

1. The tearing process consists of partitioning the original matrix in such a way that it may be expressed as the sum of two matrices. The first

of these matrices should have the property that it has sub-matrices arranged along the diagonal and zero elements elsewhere; the second must be such that the sum of all the elements in any row or column is zero. It is convenient to divide the matrix Y as follows:

$$
Y = \begin{array}{c|c|c|c|c|c|}
 & 1 & 2 & 3 & 4 & 5 \\ \hline
1 & 6 & & 4 & & \\ \hline
2 & & 2 & 1 & 2 & \\ \hline
3 & 4 & 1 & 3 & 3 & 6 \\ \hline
4 & & 2 & 3 & 3 & 2 \\ \hline
5 & & & 6 & 2 & 5 \\ \hline
\end{array}
$$

$$
= \begin{array}{c|c|c|c|c|c|}
 & 1 & 2 & 3 & 4 & 5 \\ \hline
1 & 10 & & & & \\ \hline
2 & & 5 & & & \\ \hline
3 & & & 8 & 3 & 6 \\ \hline
4 & & & 3 & 5 & 2 \\ \hline
5 & & & 6 & 2 & 5 \\ \hline
\end{array}
\; + \;
\begin{array}{c|c|c|c|c|c|}
 & 1 & 2 & 3 & 4 & 5 \\ \hline
1 & -4 & & 4 & & \\ \hline
2 & & -3 & 1 & 2 & \\ \hline
3 & 4 & 1 & -5 & & \\ \hline
4 & & 2 & & -2 & \\ \hline
5 & & & & & \\ \hline
\end{array}
$$

$$= Y_1 + y' \qquad (40)$$

Note that the off-diagonal elements of Y are first copied into either Y_1 or y'; the diagonal elements of y' are then filled in; and finally the diagonal elements of Y_1 are supplied such as to fulfill the inequality.

It is desirable to have as many zeros in y' as possible. In order to achieve this end, it may be necessary to rearrange Y by interchanging columns, and then interchanging corresponding rows (Y must remain a symmetrical matrix).

The remaining part of the first step is to invert the matrix Y_1; in order to do this, the diagonal sub-matrices may be inverted individually. It is this property of the inversion of Y_1 that makes it possible to save effort in inverting Y, or that makes it possible to invert Y at all. The result of the inversion of Y_1 is:

$$Z_1 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0.100 & & & & \\ 2 & & 0.200 & & & \\ 3 & & & 1.400 & -0.200 & -1.600 \\ 4 & & & -0.200 & 0.267 & 0.133 \\ 5 & & & -1.600 & 0.133 & 2.067 \end{array} \qquad (41)$$

2. The diagonal matrix $\underline{z}$ is formed from y' by writing along the diagonal of $\underline{z}$ the inverses of all of the off-diagonal elements of y'. For convenience in step 3, each row and column [illegible] be marked with the row and column from which the element came, as follows:

$$
z = \begin{array}{c|ccc|}
 & 1\text{-}3 & 2\text{-}3 & 2\text{-}4 \\
\hline
1\text{-}3 & -0.25 & & \\
2\text{-}3 & & -1.00 & \\
2\text{-}4 & & & -0.50 \\
\hline
\end{array} \tag{42}
$$

3. The connection matrix C is now easily determined. The vertical direction has the dimension and indices of Y; the horizontal direction the dimension and indices of z. The elements are easily determined from these indices as follows:

   a. If a row index is the same as the first part of a column index, the element of the row and column is 1.

   b. If the row index is the same as the second part of the column index, the element is -1.

   c. If the row index does not appear in the column index, the element is zero.

Following these rules, C is obtained:

$$
C = \begin{array}{c|ccc|}
 & 1\text{-}3 & 2\text{-}3 & 2\text{-}4 \\
\hline
1 & 1 & & \\
2 & & 1 & 1 \\
3 & -1 & -1 & \\
4 & & & -1 \\
5 & & & \\
\hline
\end{array} \tag{43}
$$

4. From this point the procedure is exactly the same as for the solution of a network when the

inverse is required. The matrix Z_2 is determined by

$$Z_2 = Z_1' + z = C_t Z_1 C + z$$

	1-3	2-3	2-4
1-3	1.250	1.400	-0.200
= 2-3	1.400	0.600	
2-4	-0.200		-0.033

(44)

5. Finally, the inverse Z of the matrix Y is given, as in (24), by

$$Z = Z_1 - Z_1 C Y_2 C_t Z_1$$

	1	2	3	4	5
1	0.112	-0.204	0.082	0.163	-0.163
2	-0.204	4.735	0.306	-4.388	1.388
= 3	0.082	0.306	-0.122	-0.245	0.245
4	0.163	-4.388	-0.245	4.510	-1.510
5	-0.163	1.388	0.245	-1.510	0.510

(45)

III. Application

One situation in which the concepts of Diakoptics may be very useful is that in which a portion of the network changes, the remainder being fixed. The system is to be solved for each of several sets of values of the elements in the varying part of the network. Diakoptics is useful in this situation because it allows the fixed portion of the network to be solved once for the entire problem; for each set of data only the variable network and the intersection network need to be solved.

The solution follows those outlined in Section II. In tearing the network, there are two possibilities. The first of these is that the tearing can be done in such a manner as to eliminate the variable elements from the sub-networks entirely. In this case, only the intersection network needs to be solved for each set of values of the variable elements. A network containing one variable element R is shown in Figure 11.

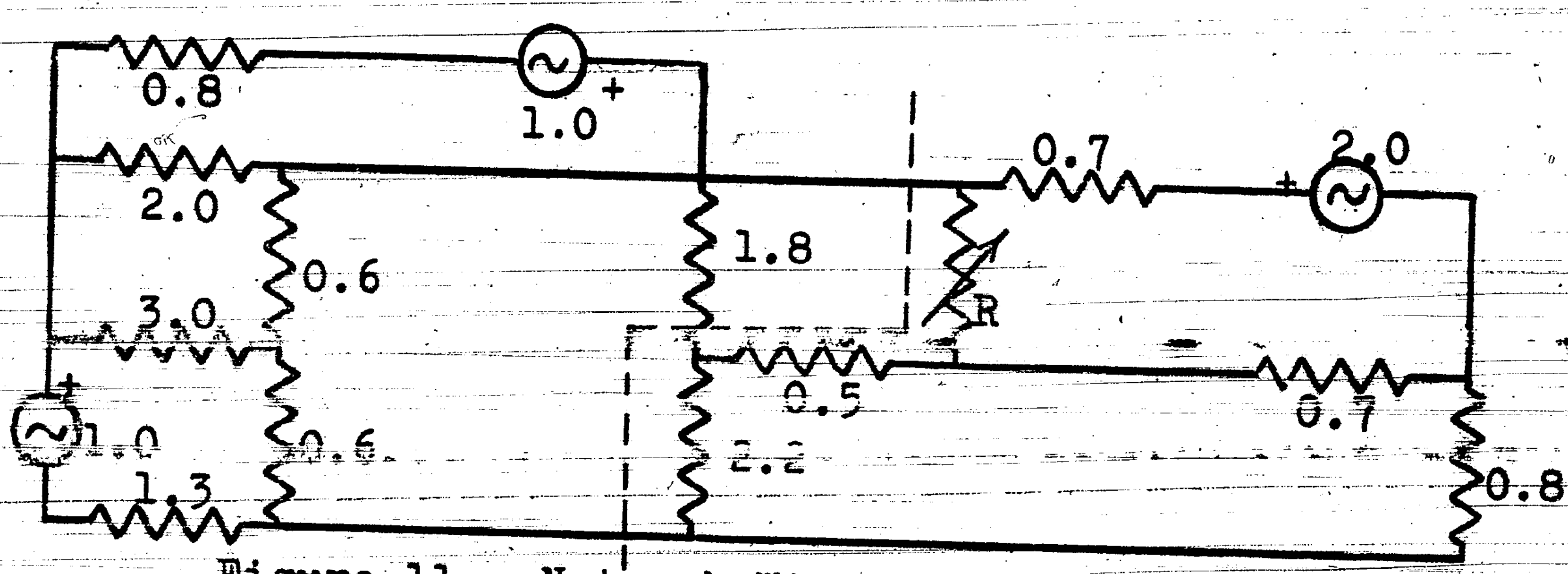


Figure 11. Network With Variable Element.

The network may also be torn so that the variable elements appear in one of the sub-networks. It is desirable that the sub-network contain as few elements as possible so that its solution will not be made unduly difficult. Figure 12 shows a line of tearing that would result in a single loop sub-network containing the variable R, along with a six loop sub-network containing the fixed elements.

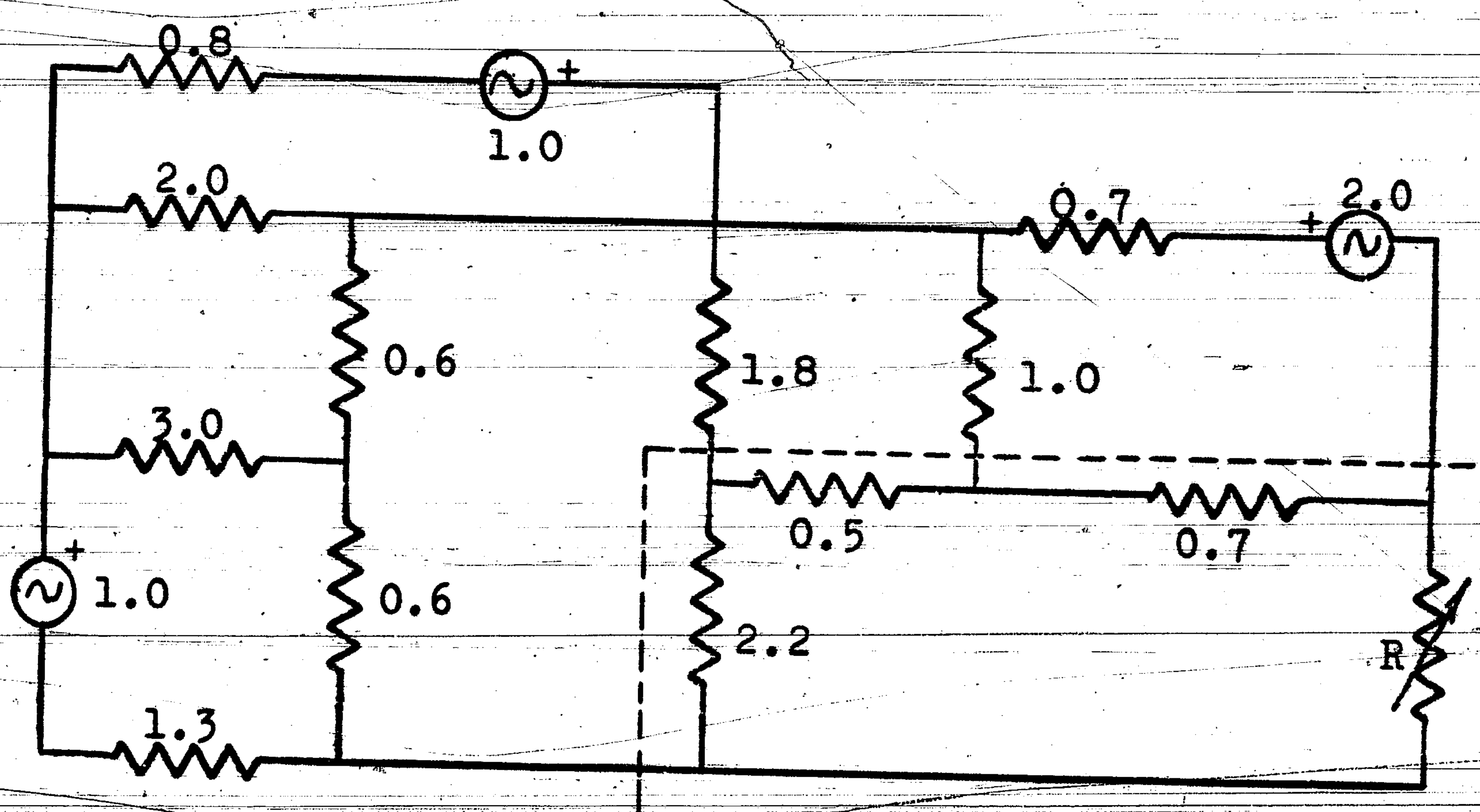
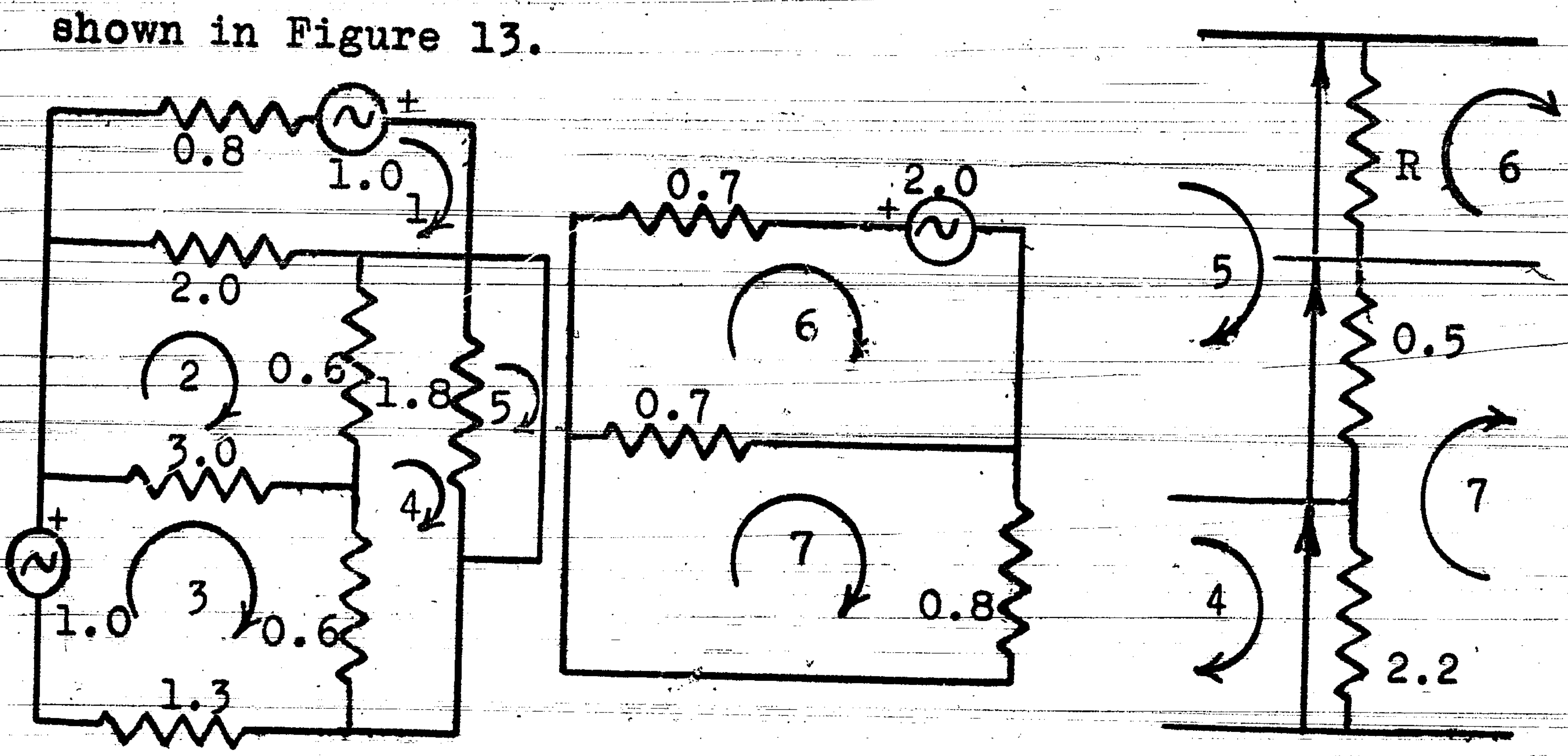


Figure 12. Network With Variable Element.

The second method of tearing will ordinarily result in more work for a network with few variable elements, while the first will be more difficult with many variable elements.

In performing the calculations, all of the calculations which involve only the fixed portion should be performed first so that there will be a minimum of calculation to be done for each set of variable elements. In order to illustrate the procedure, the circuit of Figure 11 will be solved for four values of the resistor R. The analysis is based on the sub-networks and intersection network as shown in Figure 13.



a. Sub-network A. b. Sub-network B. c. Intersection Network

Figure 13. Parts of Example Network.

The sub-network impedances are:

Z_a =

	1	2	3	4	5
1	2.8	-2.0			
2	-2.0	5.6	-3.0	-0.6	
3		-3.0	4.9	-0.6	
4		-0.6	-0.6	3.0	-1.8
5				-1.8	1.8

(46)

$$
Z_b = \begin{array}{c|cc} & 6 & 7 \\ \hline 6 & 1.4 & -0.7 \\ 7 & -0.7 & 1.5 \end{array} \qquad (47)
$$

The solutions combined into the matrix Y_1 are:

Y_1 =

	1	2	3	4	5	6	7
1	0.696	0.475	0.341	0.408	0.408		
2	0.475	0.665	0.477	0.571	0.571		
3	0.341	0.477	0.560	0.518	0.518		
4	0.408	0.571	0.518	1.378	1.378		
5	0.408	0.571	0.518	1.378	1.933		
						0.932	0.435
						0.435	0.870

(48)

The voltage matrix E is

E =

1	1.0
2	
3	1.0
4	
5	
6	-2.0
7	

(49)

Also, the connection matrix is

$$
C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc} & & \\ & & \\ & & \\ & & 1 \\ 1 & 1 & \\ -1 & & \\ & -1 & -1 \end{array}\right] \end{array}
\qquad (50)
$$

The admittance matrix $\underline{y}$ of the intersection network is

$$
y = \begin{array}{c|ccc} & a & b & c \\ \hline a & R^{-1} & & \\ b & & 2.000 & \\ c & & & 0.455 \end{array}
\qquad (51)
$$

Examination of the current equation

$$
I = Y_1E - Y_1CZ_2C_tY_1E \qquad (35)
$$

shows that the quantity

$$
Y_1E = I_1 \qquad (52)
$$

is required. This becomes

$$I_1 = \begin{array}{c|c|} 1 & 1.037 \\ 2 & 0.952 \\ 3 & 0.900 \\ 4 & 0.926 \\ 5 & 0.926 \\ 6 & -1.863 \\ 7 & -0.870 \end{array} \tag{53}$$

Other quantities that may be determined before the value of R is fixed are Y_1C and C_tY_1E. Since

$$Z_2 = (C_tY_1C + y)^{-1} = (Y_2)^{-1} \tag{54}$$

the product C_tY_1C may be found. Also, it is convenient to perform the summation in Y_2, omitting the element R^{-1} in $\underline{Y}$ temporarily. This completes that part of the solution that does not depend on R.

After selecting a value for R, and adding its inverse into the matrix Y_2, the operations that must be performed are one inversion of a three-by-three matrix, two products of two-way matrices with one-way matrices to obtain the current matrix I_2, and the subtraction of I_2 from I_1 to obtain the final result, I. The entire solution was performed on a digital computer as described in Section IV, with results for four values of R as follows:

$$R = \quad 1.000 \qquad 2.000 \qquad 0.500 \qquad 10.000$$

$$I = \begin{bmatrix} 0.688 \\ 0.463 \\ 0.456 \\ -0.254 \\ -0.556 \\ -1.181 \\ -0.396 \end{bmatrix} \quad \begin{bmatrix} 0.669 \\ 0.437 \\ 0.432 \\ -0.318 \\ -0.685 \\ -1.079 \\ -0.428 \end{bmatrix} \quad \begin{bmatrix} 0.709 \\ 0.492 \\ 0.483 \\ -0.183 \\ -0.413 \\ -1.294 \\ -0.361 \end{bmatrix} \quad \begin{bmatrix} 0.645 \\ 0.403 \\ 0.402 \\ -0.399 \\ -0.849 \\ -0.949 \\ -0.468 \end{bmatrix} \tag{55}$$

The time required to determine the currents was about the same as would have been required to solve four sets of simultaneous equations for the currents; however, after the preliminary matrices were calculated, the time for each successive current was only one-third of the time required for the solution of a set of simultaneous equations.

IV. Calculations

Calculations of results for the examples in this thesis were performed on an LGP-30 computer. This is a medium-size, binary, fixed point, stored program, punched tape input computer utilizing magnetic drum storage. All arithmetic operations were performed in a "Floating Point Interpretive Routine" which essentially is a program which allows data to be scaled optimally during calculations. Subroutines were also available to handle matrix multiplication, inversion, subtraction, and addition for numerical matrices. The programs developed specifically for the problems at hand were merely "interconnections" of these various subroutines, and one additional subroutine to provide an acceptable output format.

The program used in Section II of the thesis was developed from the flow chart shown as Chart 1. It was not designed for speed of calculation, but rather for convenience in making available the desired intermediate results. In fact, for problems of the size of the examples, no particular advantage accrues from the use of Diakoptics.

The resultant voltages were not determined by the step-by-step procedure indicated in Section II, but rather by finding Z and multiplying by I. This was done in order

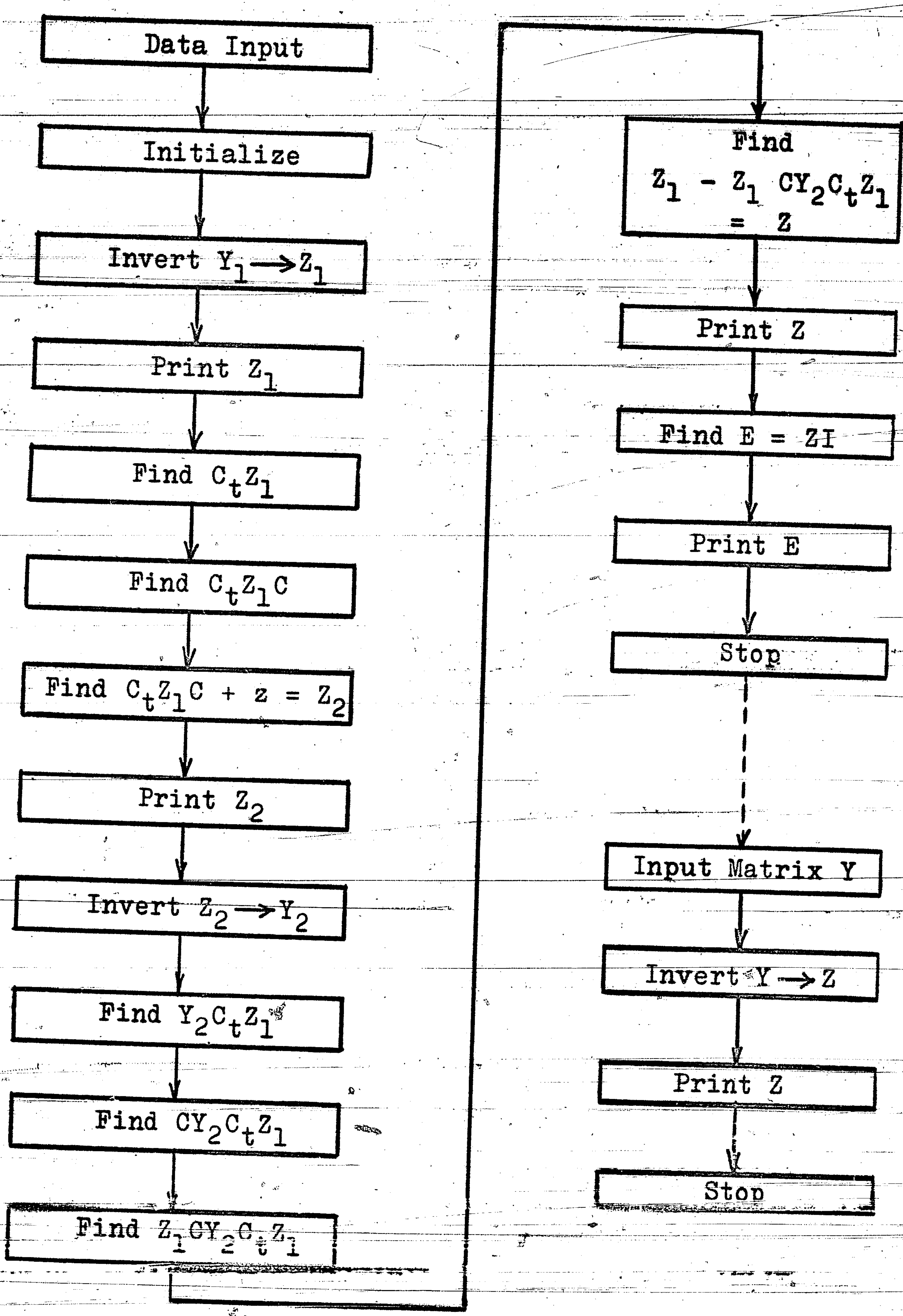


Chart 1. Flow Diagram, Program I.

to make Z available to compare against the check obtained at the end of the program.

The calculations were carried internally at the equivalent of about eight decimal places; however, the estimated accuracy of the final results is five places in most cases. The total solution time for a seven-node problem was about 27 minutes. Actual inversion time for a seven-by-seven matrix was six minutes, thirty seconds, not including the time required for print-out of results.

A second program utilizing the concepts of Diakoptics to reduce the time required for successive solutions was used for the example of Section III. The flow diagram for this program is given as Chart 2. The program was designed to operate three ways by incorporating in the program two steps where the operator might influence the operation of the program. These are indicated by the diamond-shaped blocks marked "A" and "C" on the diagram. By depressing a switch on the computer console the operator can cause the computer to take whichever branch of the program is desired. Thus, the three possibilities of operation are:

1. Normally the program will find only the currents as determined by taking the "L" branch at "A".
2. The program will find the admittance matrix Y by taking the "Y" branch at "a" and "no" at "C".

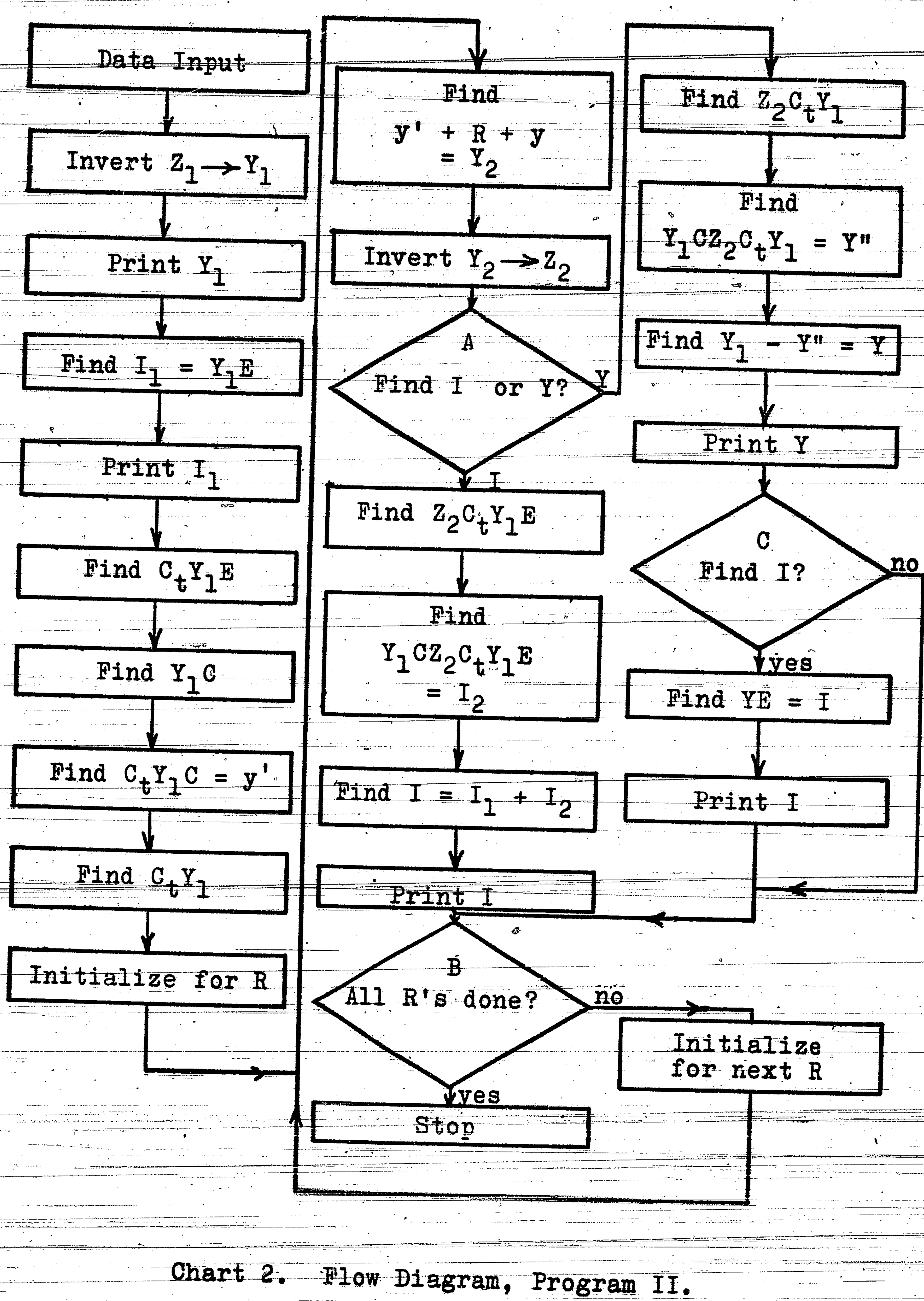


Chart 2. Flow Diagram, Program II.

3. The program will find Y, and from it compute I by taking both the "Y" branch at "A" and the "yes" branch at "C".

The box marked "B" is an internal test whose outcome is determined by a code supplied with the data.

The time required to determine the four current matrices as described in Section III, with the program operating normally, was about 22 minutes. The currents were printed out at intervals of one and one-half minutes, of which about half a minute was required for the actual printing.

V. Discussion

The primary use of Diakoptics is in the solution of large systems where the solution of a single set of simultaneous equations or the inversion of a single large matrix is difficult, time-consuming, or cannot be done at all by the conventional methods. Kron, for example, has performed the solution of a system with 256 variables[6] by tearing the system into 16 sub-systems of 16 variables each, and an intersection network, also of 16 variables. Thus, a task which was impossible otherwise with the equipment at hand was solved without great difficulty using a small computer.

A second important use is in the solution of a network several times in succession in a situation where a small portion of the network changes between solutions. The solution of such a network has been described in Section III. This idea can be extended to the solution of other systems in which whole sub-networks are replaced by other sub-networks which may be of entirely different size or even of different physical nature. This then leads to a saving of effort in that the solution of a standard network unit can be saved and used in the solution of a system including the network unit. Thus, the impedance matrix and its inverse for the network unit can be used to describe the network unit in much the same sense as the resistance of a resistor describes the resistor.

The analysis of systems to determine their response in the transient state involves setting up simultaneous differential equations. These may be solved by the use of Laplace transforms and the transformed simultaneous equations represented in matrix form. The elements of the transient state admittance or impedance will be polynomials in the complex frequency variable. Such matrices may be inverted with the aid of Diakoptics, following the same method of analysis as is used for inverting the numerical matrices of steady state analysis. However, it will be found that the process will introduce extraneous factors into the numerators and denominators which must be factored out before the inverse transforms can be taken.

The method of partitioning can also be used in solving sets of simultaneous equations and inverting matrices. For example, let a set of simultaneous equations be partitioned in the following form:

$$\begin{aligned} E_1 &= Z_1 I_1 + Z_2 I_2 \\ E_2 &= Z_3 I_1 + Z_4 I_2 \end{aligned} \qquad (56)$$

In order to solve this set of equations, it is convenient to determine the following quantities:

$$Y_1 = (Z_1)^{-1}$$
$$A = Y_1Z_2 \tag{57}$$
$$Y_4' = (Z_4 - Z_3A)^{-1}$$

Then, the resultant currents are

$$I_2 = Y_4'(E_2 - Z_3Y_1E_1)$$
$$I_1 = Y_1E_1 - AI_2 \tag{58}$$

Now, the corresponding equation using Diakoptics is

$$I = Y_1E - Y_1CZ_2C_tY_1E \tag{35}$$

where

$$Z_2 = (C_tY_1C + y)^{-1} \tag{54}$$

Note that the quantities in (35) and (54) do not have the same meanings as those in (56), (57), and (58). Following are points of comparison of the two methods of solution:

1. Both (58) and (35) may be multiplied out as successive products of two-way matrices by one-way matrices when the Y matrix is not required.
2. The equations resulting from the application of Diakoptics appear to require more multiplications than those of partitioning; however, multiplication of a matrix by C or C_t is nothing more than a process of addition or subtraction; also, C usually has a large proportion of zeros.

3. Diakoptics requires more analysis than partitioning. Partitioning may be applied to any matrix without knowledge of its zero elements. Diakoptics requires knowledge of the elements of the matrix or a graph in order to determine a line of tearing and an intersection network.

4. Diakoptics lends itself to breaking the system into more than two parts; the solution of a matrix becomes very complex if the matrix is partitioned into more than four parts.

5. Diakoptics requires that the matrix have a large proportion of zero elements; if this is not the case, the intersection network becomes so large as to become difficult to invert. The partitioning of a matrix does not depend on the presence of zero elements.

6. The analysis of a network by Diakoptics does not require setting up the entire network at one time; the network is analyzed in small parts. Partitioning requires the complete equations of the network at one time for solution.

7. The sub-networks of Diakoptics which appear in Y_1 have definite and useful significance in that they represent the physical sub-systems and may be used again; the sub-matrices and intermediate

results of partitioning have no such special significance.

7. Inversion of matrices by Diakoptics requires that the matrix be symmetrical; it is immaterial whether a matrix is symmetrical or not in partitioning.

From the above, it appears that the choice as to which method to use for a given problem depends on a consideration of several features of both methods. In general, partitioning is a method generally applicable to a wide range of problems, whereas Diakoptics is limited to a somewhat smaller range of problems for which it is more useful than partitioning.

There is another useful feature of Diakoptics. The accuracy of the final solution may be considerably improved over that obtained by standard matrix inversion due to the reduction in order of the matrices to be inverted. Another way of looking at this point is to observe that in a network, the sub-networks which are solved are generally isolated in some degree from the rest of the network because of the limited number of interconnections with the rest of the network. Thus, these sub-networks are solved individually to a degree of accuracy which is much better than the accuracy obtained by one single solution; these individual solutions are corrected using the intersection network to provide the complete solution.

Appendix

Symbols

Admittance - Y, Y_1, Y_1', Y_2, Y_a, Y_b, y

Impedance - Z, Z_1, Z_1', Z_2, Z_a, Z_b, z

Voltage - E, E_1, E_2, e

Current - I, I_1, I_2, i, i'

Resistance - R

Connection Matrix - C

Transpose of Connection Matrix - C_t

Inverse - $(Y)^{-1}$

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Biography

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